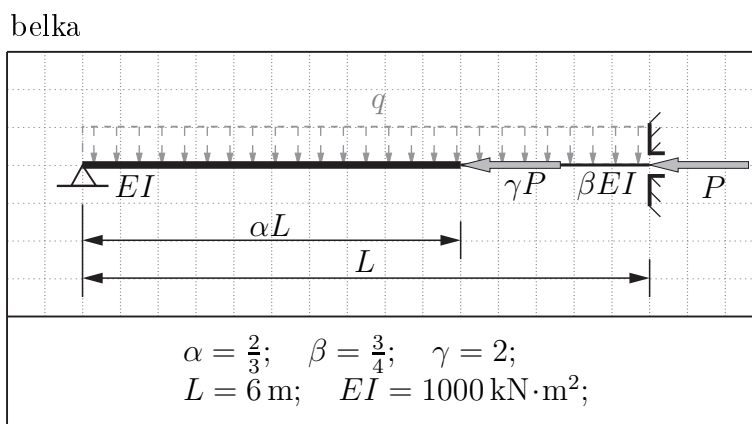


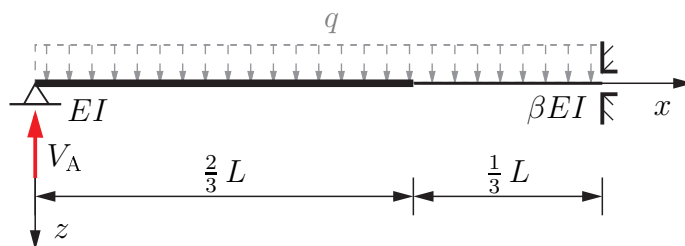
Zadanie 6. – rozwiązanie przykładowe

Dla pręta pokazanego na rysunku wyznaczyć krytyczną wartość siły P oraz współczynnik wyboczeniowy μ . Wykorzystać energetyczne kryterium Timoshenki przyjmując jako postulowaną postać wyboczenia linię ugięcia belki wyznaczoną dla zadanego obciążenia rozłożonego q .



1. Postać wyboczenia

1.1. Równanie różniczkowe osi odkształconej



$$\text{I) } 0 \leq x \leq \frac{2}{3}L$$

$$\begin{aligned} M_y^{\text{I}}(x) &= V_A x - \frac{1}{2} q x^2 \\ EI w_1''(x) &= \frac{1}{2} q x^2 - V_A x \\ EI w_1'(x) &= \frac{1}{6} q x^3 - \frac{1}{2} V_A x^2 + C_1 \\ EI w_1(x) &= \frac{1}{24} q x^4 - \frac{1}{6} V_A x^3 + C_1 x + D_1 \end{aligned}$$

$$\text{II) } \frac{2}{3}L \leq x \leq L$$

$$\begin{aligned} M_y^{\text{II}}(x) &= V_A x - \frac{1}{2} q x^2 \\ \frac{3}{4} EI w_2''(x) &= \frac{1}{2} q x^2 - V_A x \\ \frac{3}{4} EI w_2'(x) &= \frac{1}{6} q x^3 - \frac{1}{2} V_A x^2 + C_2 \\ \frac{3}{4} EI w_2(x) &= \frac{1}{24} q x^4 - \frac{1}{6} V_A x^3 + C_2 x + D_2 \end{aligned}$$

1.2. Wyznaczenie stałych z warunków brzegowych

$$\left. \begin{aligned} (1) \quad w_1(0) &= 0 & \Rightarrow D_1 &= 0 \\ (2) \quad w_1'\left(\frac{2}{3}L\right) &= w_2'\left(\frac{2}{3}L\right) & \Rightarrow C_1 - \frac{4}{3}C_2 + \frac{2}{27}V_AL &= \frac{4}{243}qL^3 \\ (3) \quad w_1\left(\frac{2}{3}L\right) &= w_2\left(\frac{2}{3}L\right) & \Rightarrow \frac{2}{3}C_1L - \frac{8}{9}C_2L - \frac{4}{3}D_2 + \frac{4}{243}V_AL &= \frac{2}{729}qL^4 \\ (4) \quad w_2'(L) &= 0 & \Rightarrow C_2 - \frac{1}{2}V_AL^2 &= -\frac{1}{6}qL^3 \\ (5) \quad w_2(L) &= 0 & \Rightarrow C_2L + D_2 - \frac{1}{6}V_AL^3 &= -\frac{1}{24}qL^4 \end{aligned} \right\} \Rightarrow \left\{ \begin{aligned} C_1 &= \frac{136}{6075}qL^3 \\ C_2 &= \frac{31}{1200}qL^3 \\ D_1 &= 0 \\ D_2 &= -\frac{1}{300}qL^4 \\ V_A &= \frac{77}{200}qL \end{aligned} \right.$$

$$w_1(x) = \frac{1}{EI} \left[\frac{1}{24} q x^4 - \frac{1}{6} \cdot \frac{77}{200} qL x^3 + \frac{136}{6075} qL^3 x \right]$$

$$w_2(x) = \frac{4}{3EI} \left[\frac{1}{24} q x^4 - \frac{1}{6} \cdot \frac{77}{200} qL x^3 + \frac{31}{1200} qL^3 x - \frac{1}{300} qL^4 \right]$$

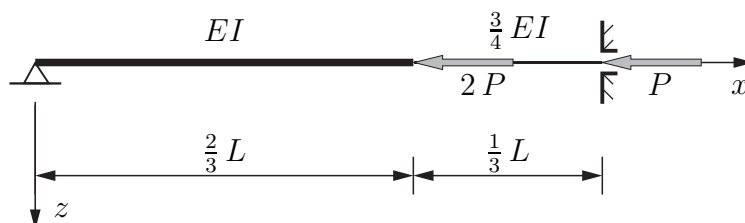
1.3. Rysunek postaci odkształconej belki



2. Obciążenie krytyczne

2.1. Kryterium energetyczne Timoshenki

$$\int_L EI (w'')^2 dx + \int_L N (w')^2 dx = 0$$



(I) $x \in \left\langle 0, \frac{2}{3}L \right\rangle$ $w'_I(x) = \frac{1}{EI} \left(\frac{1}{6} q x^3 - \frac{77}{400} q L x^2 + \frac{136}{6075} q L^3 \right)$ $w''_I(x) = \frac{1}{EI} \left(\frac{1}{2} q x^2 - \frac{77}{200} q L x \right)$ $N_I(x) = -3P$ $(EI)_I = EI$	(II) $x \in \left\langle \frac{2}{3}L, L \right\rangle$ $w'_{II}(x) = \frac{4}{3EI} \left(\frac{1}{6} q x^3 - \frac{77}{400} q L x^2 + \frac{31}{1200} q L^3 \right)$ $w''_{II}(x) = \frac{4}{3EI} \left(\frac{1}{2} q x^2 - \frac{77}{200} q L x \right)$ $N_{II}(x) = -P$ $(EI)_{II} = \frac{3}{4}EI$
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$$\underbrace{\int_0^{\frac{2}{3}L} EI (w''_I)^2 dx}_{I_1} + \underbrace{\int_{\frac{2}{3}L}^L \frac{3}{4} EI (w''_{II})^2 dx}_{I_2} - P \underbrace{\int_0^{\frac{2}{3}L} 3 (w'_I)^2 dx}_{I_3} - P \underbrace{\int_{\frac{2}{3}L}^L (w'_{II})^2 dx}_{I_4} = 0$$

$$I_1 + I_2 - P(I_3 + I_4) = 0$$

2.2. Wartość krytyczna siły P

$$P_{kr} = \frac{I_1 + I_2}{I_3 + I_4}$$

$$\begin{aligned} I_1 &= \int_0^{\frac{2}{3}L} EI (w''_I)^2 dx = \frac{q^2}{EI} \int_0^{\frac{2}{3}L} \left(\frac{1}{2} x^2 - \frac{77}{200} L x \right)^2 dx = \frac{q^2}{EI} \int_0^{\frac{2}{3}L} \left(\frac{1}{4} x^4 - \frac{77}{200} L x^3 + \frac{77^2}{200^2} L^2 x^2 \right) dx = \\ &= \frac{q^2}{EI} \left(\frac{1}{20} x^5 - \frac{77}{800} L x^4 + \frac{5929}{120000} L^2 x^3 \right) \Big|_0^{\frac{2}{3}L} = \frac{2687}{1215000} \frac{q^2 L^5}{EI} = 2,212 \cdot 10^{-3} \frac{q^2 L^5}{EI} \end{aligned}$$

$$\begin{aligned}
 I_2 &= \int_{\frac{2}{3}L}^L \frac{3}{4} EI (w''_{II})^2 dx = \frac{4q^2}{3EI} \int_{\frac{2}{3}L}^L \left(\frac{1}{2} x^2 - \frac{77}{200} Lx \right)^2 dx = \\
 &= \frac{4q^2}{3EI} \left(\frac{1}{20} x^5 - \frac{77}{800} Lx^4 + \frac{5929}{120000} L^2 x^3 \right) \Big|_{\frac{2}{3}L}^L = \frac{9203}{7290000} \frac{q^2 L^5}{EI} = 1,262 \cdot 10^{-3} \frac{q^2 L^5}{EI}
 \end{aligned}$$

$$\begin{aligned}
 I_3 &= \int_0^{\frac{2}{3}L} 3 (w'_I)^2 dx = \frac{3q^2}{EI^2} \int_0^{\frac{2}{3}L} \left(\frac{1}{6} x^3 - \frac{77}{400} Lx^2 + \frac{136}{6075} L^3 \right)^2 dx = \\
 &= \frac{3q^2}{EI^2} \int_0^{\frac{2}{3}L} \left(\frac{1}{36} x^6 - \frac{77}{1200} Lx^5 + \frac{5929}{160000} L^2 x^4 + \frac{136}{18225} L^3 x^3 - \frac{1309}{151875} L^4 x^2 + \frac{18496}{36905625} L^6 \right) dx = \\
 &= \frac{3q^2}{EI^2} \left(\frac{1}{252} x^7 - \frac{77}{7200} Lx^6 + \frac{5929}{800000} L^2 x^5 + \frac{34}{18225} L^3 x^4 - \frac{1309}{455625} L^4 x^3 + \frac{18496}{36905625} L^6 x \right) \Big|_0^{\frac{2}{3}L} = \\
 &= 3,621 \cdot 10^{-4} \frac{q^2 L^7}{EI^2}
 \end{aligned}$$

$$\begin{aligned}
 I_4 &= \int_{\frac{2}{3}L}^L (w'_{II})^2 dx = \frac{16q^9}{EI^2} \int_{\frac{2}{3}L}^L \left(\frac{1}{6} x^3 - \frac{77}{400} Lx^2 + \frac{31}{1200} L^3 \right)^2 dx = \\
 &= \frac{16q^2}{9EI^2} \int_{\frac{2}{3}L}^L \left(\frac{1}{36} x^6 - \frac{77}{1200} Lx^5 + \frac{5929}{160000} L^2 x^4 + \frac{31}{3600} L^3 x^3 - \frac{2387}{240000} L^4 x^2 + \frac{961}{1440000} L^6 \right) dx = \\
 &= \frac{16q^2}{9EI^2} \left(\frac{1}{252} x^7 - \frac{77}{7200} Lx^6 + \frac{5929}{800000} L^2 x^5 + \frac{31}{14400} L^3 x^4 - \frac{2387}{720000} L^4 x^3 + \frac{961}{1440000} L^6 x \right) \Big|_{\frac{2}{3}L}^L = \\
 &= 5,820 \cdot 10^{-5} \frac{q^2 L^7}{EI^2}
 \end{aligned}$$

$$P_{kr} = \frac{(2,212 \cdot 10^{-3} + 1,262 \cdot 10^{-3}) \frac{q^2 L^5}{EI}}{(3,621 \cdot 10^{-4} + 5,820 \cdot 10^{-5}) \frac{q^2 L^7}{EI^2}} = 8,265 \frac{EI}{L^2} = 8,265 \frac{1000 \text{ kN m}^2}{6^2 \text{ m}^2} = 229,6 \text{ kN}$$

$$\mu = \sqrt{\frac{\pi^2 EI}{P_{kr} L^2}} = \sqrt{\frac{\pi^2}{8,265}} = 1,093$$